AN INTRODUCTION TO POINT-SET TOPOLOGY: $\mathbb{S}^1$ AS A COMPACT HAUSDORFF TOPOLOGICAL SPACE

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Abstract. We introduce point-set topology as a generalisation of analysis on $\mathbb{R}$. As background, we assume knowledge that is usually covered in the first few chapters of an introductory analysis text such as Spivak's *Calculus* (the idea of limits and continuity) and some familiarity with set operations. The aim is to span a typical first course in topology with a cohesive subset of what might normally be discussed in a full semester of lectures. The development of $\mathbb{S}^1$ as a compact Hausdorff space will be a central motivating theme.

Contents

1. Introduction

The point of point-set topology (hereafter referred to simply as topology) is to abstract analysis on $\mathbb{R}$ (or $\mathbb{R}^n$ or $\mathbb{C}^n$). Recall that analysis in $\mathbb{R}$ studies what it means for two objects (points, sequences, functions) to be close to one another: the definition of pointwise continuous functions in $\mathbb{R}$ says that a function f is continuous iff we can guarantee $f(x)$ will be arbitrarily close to $f(x_0)$ by restricting x to be sufficiently close to x_0.

It might seem that the idea of closeness inherently requires some definition of distance so we will briefly discuss metric spaces (which abstract the notion of distance). However, once we have defined topological space, we will see that we can sensibly decide which points in a set are "close" to each other by specifying what the "open" subsets are. So long as our prescription for the "open" sets is consistent with the definition of a topology, we will be able to do analysis on more general sets without reference to distance. Example 3.1.4 gives a context in which the set $\{\text{Alice}, \text{Bob}\}$ is "open". To emphasise this point, I will use the non-standard naming convention of referring to open sets defined by a metric as being metric-open and those from a topology simply as open.

Analysis without reference to distance is precisely what we need to be able to consider "rubber sheet geometry" which is studied under the name algebraic topology (specifically homotopy theory and the fundamental group) and addresses questions such as the difference between the torus (doughnut) and the sphere. Algebraic topology is typically treated in a second course on topology and will not be studied here. For an introduction to Algebraic Topology, readers are referred to [Hat02] or [Die08].

2. Metric Spaces

Metric spaces allow us to generalise analysis on $\mathbb{R}$ by first abstracting what it means to be a "distance". Since topological spaces allow us to generalise analysis on metric spaces and this is an introduction to topology, we shall discuss metric spaces only briefly — providing minimal definitions and few examples. For a more thorough treatment of metric spaces, see e.g. [Sut10].

2.0.1. **Definition.** *Metric space.* A metric space is an ordered pair (X, d) where X is a set (the space) and d is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$:

(i) $d(x, y) \geq 0$ (non-negativity)
(ii) $d(x, y) = 0 \iff x = y$ (identity of indiscernibles)
(iii) $d(x, y) = d(y, x)$ (symmetry)
(iv) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

2.0.2. **Example.** A metric space on $\mathbb{R}$. We choose the metric $d_1(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$ where $|x|$ is the usual modulus sign. We verify:

(i) $|z| > 0$ for all $z \in \mathbb{R}$ and $\mathbb{R}$ is closed under addition
(ii) $|x-y| = 0 \iff x-y = 0 \iff x = y$ by trichotomy of $\mathbb{R}$ (if $x < y$ or $y > x$ then we obtain a contradiction so $x = y$)
(iii) If $x < y$ then $|x - y| = -(x - y) = (y - x) = |y - x|$. The case $y < x$ follows by relabeling the previous case and if $x = y$ then $|x-y| = 0 = |y-x|$. By trichotomy of $\mathbb{R}$ we have exhausted all possibilities.
(iv) Let $x - z = A, x - y = B, y - z = C$ and notice $A = B + C$. Now consider $|A + B|^2 = (A + B)^2 = A^2 + 2AB + B^2 \leq (|A| + |B|)^2$ since $2AB \leq |2AB|$. This implies $|A + B| \leq (|A| + |B|)$ since both sides are non-negative.

2.0.3. **Example.** Some metric spaces from $\mathbb{R}^2$. We may choose the metric $d_1^2(\vec{x}, \vec{y}) = |x_1 - y_1| + |x_2 - y_2|$ and demonstrate that this is indeed a metric in a similar fashion to the above discussion of $(\mathbb{R}, d_1)$. However a more natural metric to use in this case is the Euclidean metric $d_2(\vec{x},\vec{y}) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$ which emphasises the idea that metrics define distance on a set. Proving the triangle inequality in this case takes more effort and relies on the Cauchy-Schwarz inequality. For the sake of brevity we will simply cite [Men03, Ch. 2, Thm. 2.5, Ch. 2, Sec 8] and claim $(\mathbb{R}^2, d_2)$ is a metric space. Notice that a single set may have more than one metric associated with it — it is the ordered pairs $(\mathbb{R}^2, d_1^2)$ and $(\mathbb{R}^2, d_2)$ that define the metric spaces. However, when it is clear what metric we are using, it is common to refer to the set e.g. $\mathbb{R}^2$ synonymously with the pair.

2.0.4. **Definition.** *Open ball.* Let (X, d) be a metric space. For any $x \in X, r \in \mathbb{R}^+$ we define $B_r(x) := \{y \in X : d(x,y) < r\}$ to be the open ball of radius r centred at x.

2.0.5. **Definition.** *Metric-open set.* A set $O \subset X$ is metric-open if for every $x \in O, \exists \delta > 0$ such that $B_\delta(x) \subseteq O$. This says that for a set O to be open, every element of O must have a little bit of space also in O that surrounds the element in all directions — one can "wiggle" each point a small amount whilst remaining in O.

2.0.6. **Example.** Open intervals in $(\mathbb{R}, d_1)$ are metric-open. We define the open interval (a,b) as being the set $\{x \in \mathbb{R} : a < x < b\}$. For any $x \in (a,b)$ then $x-a > 0$ and $b-x > 0$. Let $\epsilon = \min(x-a, b-x)$ and consider the open ball $B_\epsilon(x) := \{y \in \mathbb{R} : |x-y| < \epsilon\}$. Now for any $y \in B_\epsilon(x)$ then $y - x \leq |x-y| < \epsilon$ so $y < \epsilon + x \leq b - x + x = b$. Likewise $x - y \leq |x-y| < \epsilon$ so $y > x - \epsilon \geq x - (x-a) = a$. So $B_\epsilon(x) \subseteq (a,b)$. Since x was arbitrary we may find such an open ball for any $x \in (a,b)$ and conclude (a,b) is open for any $a, b \in \mathbb{R}, a < b$.

2.0.7. **Example.** Closed intervals in $(\mathbb{R}, d_1)$ are not metric-open. We define the closed interval $[a,b]$ as being the set $\{x \in \mathbb{R} : a \leq x \leq b\}$. Consider the open ball $B_\delta(a) := \{y \in \mathbb{R} : |a-y| < \delta\}$ with $\delta > 0$. Then $|a - (a - \frac{\delta}{2})| = |\frac{\delta}{2}| < \delta$ so $(a - \frac{\delta}{2}) \in B_\delta(a)$ but $(a - \frac{\delta}{2}) \notin [a,b]$ therefore $B_\delta(a) \not\subseteq [a,b]$. Since δ was arbitrary, there are no open balls around a and $[a,b]$ is not open for any $a, b \in \mathbb{R}, a \leq b$.

2.0.8. **Example.** Singleton sets in $(\mathbb{R}, d_1)$ are not metric-open. This is a special case of example 2.0.7 when $a = b$.

Note that whether or not a subset $O \subseteq X$ is metric-open depends on the metric associated with X.

2.0.9. **Example.** Singleton sets in $(\mathbb{R}, \tilde{d})$ are metric-open for some choice of $\tilde{d}$. In particular choose:

$$\tilde{d}(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Now consider the singleton set $\{x_0\}$ and the open ball $B_\delta(x_0) := \{y \in \mathbb{R} : \tilde{d}(x_0, y) < \delta\}$ with $1 > \delta > 0$. We know that $x_0 \in B_\delta(x_0)$ since $\tilde{d}(x_0, x_0) = 0 < \delta$. Now consider $y \neq x_0$ then $\tilde{d}(x_0, y) = 1 > \delta$ so $y \notin B_\delta(x_0)$. Therefore $B_\delta(x_0) = \{x_0\} \subseteq \{x_0\}$. There are no other points in $\{x_0\}$ to consider so we are done.

2.0.10. **Example.** Open balls are metric-open. Let (X, d) be a metric space and consider an open ball $B_\epsilon(x)$ for some $\epsilon > 0$ and consider $y \in B_\epsilon(x)$. If $y = x$ then $B_\epsilon(y) \subseteq B_\epsilon(x)$. If $y \neq x$ then $\delta = |x - y| < \epsilon$. Consider a point z in the open ball $B_{(\epsilon-\delta)}(y)$. By the triangle inequality, $d(x, z) \leq d(x, y) + d(y, z) < \delta + (\epsilon - \delta) = \epsilon$ so $B_{(\epsilon-\delta)}(y) \subset B_\epsilon(x)$. Then $B_\epsilon(x)$ is metric-open.

3. Topology

3.0.1. **Definition.** *Topological space.* A topological space is a set X (the space) together with a set $\mathcal{U} \subseteq \mathcal{P}(X)$ (the topology) such that:

(i) $\emptyset, X \in \mathcal{U}$ (contains the empty set and the space)
(ii) If $V_\alpha \in \mathcal{U}$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} V_\alpha \in \mathcal{U}$ (closure under arbitrary union)
(iii) If $V_1, \ldots, V_n \in \mathcal{U}$, then $\bigcap_{i=1}^{n} V_i \in \mathcal{U}$ (closure under finite intersections)

The elements of X are called *points* and the elements of $\mathcal{U}$ are called *open sets* of X. We denote a topological space $(X, \mathcal{U})$ — the ordered pair consisting of the space X and the topology $\mathcal{U}$. We refer to such a $\mathcal{U}$ as a *topology on* X. As with metric spaces, when the meaning is clear, it is common to refer to the set X synonymously with the associated topological space $(X, \mathcal{U})$.

It is important to note that the *open sets* in a topological space are defined as being exactly the members of the topology $\mathcal{U}$. These will not necessarily be the same as the open sets defined in terms of a metric (when X has some associated metric). However, each metric space does have a natural topology associated with it: the *induced topology.*

3.1. Constructing Topologies.

3.1.1. **Definition.** *Induced topology.* Let (X, d) be a metric space. Then the collection $\mathcal{O}$ consisting of all metric-open sets of X (defined by d) is called an induced topology or *the topology induced by* d.

3.1.2. **Proposition.** *An induced topology is a topology.*

Proof.

(i) Consider the empty set $\emptyset$. For each of its elements there exists an open ball which is a subset of the empty set... ..because it has no elements! So $\emptyset$ is metric-open. Now consider the entire space X. Since open balls are defined to be subsets of the space, any open ball $B_\delta(x)$ around a point $x \in X$ will be contained in X. That is to say, we may choose any $\delta > 0$. Then $B_\delta(x) \subseteq X$ for any $x \in X$. So X is metric-open. Note that we are guaranteed $x \in B_\delta(x) \neq \emptyset$ since $x \in X$ (so long as $X \neq \emptyset$) and $d(x,x) = 0 < \delta$.

(ii) If V_α is metric-open for all $\alpha \in A$ then for any x in some V_α there exists $\delta > 0$ such that $B_\delta(x) \subseteq V_\alpha$. Now consider $z \in \bigcup_{\alpha \in A} V_\alpha \implies \exists \alpha$ s.t. $z \in V_\alpha$. Then there exists $\delta > 0$ such that $B_\delta(z) \subseteq V_\alpha \subseteq \bigcup_{\alpha \in A} V_\alpha$. So $\bigcup_{\alpha \in A} V_\alpha$ is metric-open.

(iii) If V_i is metric-open for all $1 \leq i \leq n$ then for any x in some V_i there exists $\delta > 0$ such that $B_\delta \subseteq V_i$. Now consider $z \in \bigcap_{i=1}^n V_i \implies z \in V_i$ for all $1 \leq i \leq n$. Then for all $1 \leq i \leq n$ there exists $\delta_i > 0$ such that $B_{\delta_i}(z) \subseteq V_i$. Since n is finite we may consider the smallest δ_i and label it δ^*. Then $B_{\delta^*}(z) \subseteq B_{\delta_i}(z) \subseteq V_i$ for all $1 \leq i \leq n$. Then $B_{\delta^*}(z) \subseteq \bigcap_{i=1}^n V_i$ so $\bigcap_{i=1}^n V_i$ is metric-open.

$\square$

3.1.3. **Example.** The metric d_1 in $(\mathbb{R}, d_1)$ induces a topology. In example 2.0.2 we showed that $(\mathbb{R}, d_1)$ with $d_1(x,y) = |x-y|$ is a bona fide metric space. So we may apply proposition 3.1 and conclude that $\mathcal{O} = \{U \in \mathcal{P}(\mathbb{R}) : U \text{ is metric-open}\}$ is a topology and $(\mathbb{R}, \mathcal{O})$ is a topological space.

In example 2.0.6 we showed that any open interval (a,b) is metric-open in $(\mathbb{R}, d_1)$ so $(a,b) \in \mathcal{O}$ for all $a < b$. Then consider $(-\frac{1}{n}, \frac{1}{n}) \in \mathcal{O}$ for all $n \in \mathbb{N}$. Now take $\bigcap_{n \in \mathbb{N}}(-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R}$. This is equal to the singleton set $\{0\}$. Proof: $0 \in \bigcap_{n \in \mathbb{N}}(-\frac{1}{n}, \frac{1}{n})$ since for any $n \in \mathbb{N}$ then $-\frac{1}{n} < 0 < \frac{1}{n}$. Also, $z \notin \bigcap_{n \in \mathbb{N}}(-\frac{1}{n}, \frac{1}{n})$ for any $z \neq 0$ in $\mathbb{R}$ since for any $z \neq 0$ in $\mathbb{R}$ there exists some n in $\mathbb{N}$ such that $\frac{1}{n} < |z|$ by the Archimedean property. As shown in example 2.0.8, singleton sets are not metric-open in $(\mathbb{R}, d_1)$ so $\{0\} \notin \mathcal{O}$. This motivates part (iii) of the definition of a topology, which only requires closure under finite intersections — taking non-finite intersections often leaves points with no "room to wiggle".

3.1.4. **Example.** The *course topology* $\mathcal{C} = \{\emptyset, X\} \subset \mathcal{P}(X)$ is a topology:

(i) $\emptyset, X \in \mathcal{C}$

(ii) If $V_\alpha \in \mathcal{C}$ then either $V_\alpha = \emptyset$ for all $\alpha \in A$ so $\bigcup_{\alpha \in A} V_\alpha = \emptyset \in \mathcal{C}$ or $V_\alpha = X$ for some $\alpha \in A$ so $\bigcup_{\alpha \in A} V_\alpha = X \in \mathcal{C}$.

(iii) If $V_1 \dots V_n \in \mathcal{C}$ then either $V_i = X$ for all $1 \leq i \leq n$ so $\bigcap_{i=1}^n V_i = X \in \mathcal{C}$ or $V_i = \emptyset$ for some $1 \leq i \leq n$ so $\bigcap_{i=1}^n V_i = \emptyset \in \mathcal{C}$

The course topology is an example of topologically open sets which are not defined in terms of metric-open sets. For instance let $X = \{\text{Alice}, \text{Bob}\}$. Then our course topology $\mathcal{C} = \{\emptyset, \{\text{Alice}, \text{Bob}\}\}$. That is to say $\{\text{Alice}, \text{Bob}\}$ is an open set!

One of the windfalls from abstraction is generalisation. So topologically open sets should allow us to do more than metric-open sets. The course topology may seem silly — and we will see some more interesting examples in subsections 3.2 and 3.3 — but the following remark should persuade the reader that the course topology is not "trivial" (unimportant).

3.1.5. **Remark.** *The course topology is the smallest topology.*

Consider a topological space $(X, \mathcal{U})$ and the course topology $\mathcal{C} = \{\emptyset, X\}$. Then $\mathcal{C} \subseteq \mathcal{U}$ by definition of $\mathcal{U}$ as a topology since axiom (i) tells us that $\emptyset, X \in \mathcal{U}$. In particular, whenever we prove that some set $\mathcal{U}$ is a topology of X, we should be wary that it might only contain the elements $\emptyset$ and X.

3.1.6. **Definition.** *Subspace topology.* Let $(X, \mathcal{U})$ be a topological space and $Y \subseteq X$. Then we call the set $\mathcal{S} := \{V \in \mathcal{P}(Y) : \exists U \in \mathcal{U} \text{ s.t. } U \cap Y = V\}$ a subspace topology on Y.

3.1.7. **Proposition.** *A subspace topology is a topology*

Proof.

(i) $\emptyset \in \mathcal{U}$ and $\emptyset \cap Y = \emptyset$ so $\emptyset \in \mathcal{S}$. Similarly $X \in \mathcal{U}$ and $X \cap Y = Y$ so $Y \in \mathcal{S}$

(ii) Let $V_\alpha \in \mathcal{S}$ for all $\alpha \in A$. Then there exists $U_\alpha \in \mathcal{U}$ such that $V_\alpha = U_\alpha \cap Y$ for all $\alpha \in A$. Now consider $\bigcup_{\alpha \in A} V_\alpha = \bigcup_{\alpha \in A}(U_\alpha \cap Y) = (\bigcup_{\alpha \in A} U_\alpha) \cap Y$. Since $\mathcal{U}$ is a topology then $(\bigcup_{\alpha \in A} U_\alpha) \in \mathcal{U}$ so $\bigcup_{\alpha \in A} V_\alpha \in \mathcal{S}$

(iii) Let $V_1, \dots, V_n \in \mathcal{S}$. Then there exists $U_i \in \mathcal{U}$ such that $V_i = U_i \cap Y$ for all $1 \leq i \leq n$. Now consider $\bigcap_{i=1}^n V_i = \bigcap_{i=1}^n (U_i \cap Y) = (\bigcap_{i=1}^n U_i) \cap Y$. Since $\mathcal{U}$ is a topology then $(\bigcap_{i=1}^n U_i) \in \mathcal{U}$ so $\bigcap_{i=1}^n V_i \in \mathcal{S}$

□

3.1.8. **Example.** $[-1, 1]$ with the subspace topology from the induced topology of $(\mathbb{R}, d_1)$. These is exactly the set of all metric-open (under the d_1 metric) subsets of $\mathbb{R}$ each intersected with $[-1, 1]$. A great many of these intersections produce $\emptyset \subset [-1, 1]$.

3.1.9. **Definition.** *Product topology.* Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be topological spaces. Then we say subset $Z \subseteq X \times Y := \{(x,y) : x \in X, y \in Y\}$ is open iff for every ordered pair $(x_0, y_0) \in Z$ there exist sets $U_{x_0} \in \mathcal{U}$ and $V_{y_0} \in \mathcal{V}$ with $x_0 \in U_{x_0}$ and $y_0 \in V_{y_0}$ such that $U_{x_0} \times V_{y_0} := \{(x,y) : x \in U_{x_0}, y \in V_{y_0}\} \subseteq Z$. The collection of all such Z forms the product topology $\mathcal{K}_{X\times Y}$.

Note that we only require such sets U_{x_0} and V_{y_0} to exist for each $(x_0, y_0) \in Z$ — we will not in general be able to find sets U and V which satisfy this condition for all $(x,y) \in Z$ simultaneously.

3.1.10. **Proposition.** *A product topology $\mathcal{K}_{X\times Y}$ is a topology.*

Proof.

(i) The empty set $\emptyset$ is in $\mathcal{K}_{X\times Y}$ vacuously since there are no ordered pairs in $\emptyset$. For any ordered pair $(x_0, y_0) \in X \times Y$ then $x_0 \in X$ and $y_0 \in Y$. We know $X \in \mathcal{U}$ and $Y \in \mathcal{V}$ and also $X \times Y \subseteq X \times Y$ so $X \times Y \in \mathcal{K}_{X\times Y}$.

(ii) Let $K_\alpha \in \mathcal{K}_{X\times Y}$ for all $\alpha \in A$. Consider $(x_0, y_0) \in \bigcup_{\alpha\in A} K_\alpha$. Then $(x_0, y_0) \in K_\alpha$ for some $\alpha \in A$ so there exists $U_{x_0} \in \mathcal{U}$ and $V_{y_0} \in \mathcal{V}$ with $x_0 \in U_{x_0}$ and $y_0 \in V_{x_0}$ such that $U_{x_0} \times V_{y_0} \subseteq K_\alpha \subseteq \bigcup_{\alpha\in A} K_\alpha$. So $\bigcup_{\alpha\in A} K_\alpha \in \mathcal{K}_{X\times Y}$.

(iii) Let $K_1, \ldots, K_n \in \mathcal{K}_{X\times Y}$. Consider $(x_0, y_0) \in \bigcap_{i=1}^n K_i$. Then $(x_0, y_0) \in K_i$ for all $1 \le i \le n$ so there exists $U_{i,x_0} \in \mathcal{U}$ and $V_{i,y_0} \in \mathcal{V}$ with $x_0 \in U_{i,x_0}$ and $y_0 \in V_{i,y_0}$ for all $1 \le i \le n$ such that $U_{i,x_0} \times V_{i,y_0} \subseteq K_i$. Now consider $(z_1, z_2) \in (\bigcap_{i=1}^n U_{i,x_0}) \times (\bigcap_{i=1}^n V_{i,y_0})$ implies $z_1 \in U_{i,x_0}$ and $z_2 \in V_{i,y_0}$ for all $1 \le i \le n$. Then $(z_1, z_2) \in U_{i,x_0} \times V_{i,y_0} \subseteq K_i$ for all $1 \le i \le n$ so $(z_1, z_2) \in \bigcap_{i=1}^n K_i$. We conclude that $(\bigcap_{i=1}^n U_{i,x_0}) \times (\bigcap_{i=1}^n V_{i,y_0}) \subseteq \bigcap_{i=1}^n K_i$. Since $(\bigcap_{i=1}^n U_{i,x_0}) \in \mathcal{U}$ and $(\bigcap_{i=1}^n V_{i,y_0}) \in \mathcal{V}$ and $(x_0, y_0) \in (\bigcap_{i=1}^n U_{i,x_0}) \times (\bigcap_{i=1}^n V_{i,y_0})$ we are done.

$\square$

3.1.11. **Example.** Take $[-1,1]$ with the subspace topology given in example 3.1.8 and form the product topology on $[-1,1] \times [-1,1] \equiv [-1,1]^2$. Recall that open sets in $[-1,1]$ were exactly the metric-open sets from $(\mathbb{R}, d_1)$ each intersected with $[-1,1]$. So for the product product topology, each point (x_0, y_0) in an open set of $[-1,1] \times [-1,1]$ is contained in the direct product of two metric-open sets from $(\mathbb{R}, d_1)$ (each intersected with $[-1,1]$).

We constructed an induced topology on $\mathbb{R}$ in example 3.1.3 and from this we derived a subspace topology on $[-1,1]$ in example 3.1.8. Above (in example 3.1.11) we formed a product topology on $[-1,1]^2$. We may now venture one step further and (finally) describe $\mathbb{S}^1$. [1]

3.1.12. **Example.** $\mathbb{S}^1$ as a subspace topology of $[-1,1]^2$. We define $\mathbb{S}^1 := \{(x,y) \in [-1,1]^2 : x^2+y^2 = 1\} \subset [-1,1]^2$. So we may endow it with a subspace topology derived from example 3.1.11.

Once we have a collection of open sets (the topology), we may use them to define *closed sets*:

3.1.13. **Definition.** *Closed set.* Let $(X,\mathcal{U})$ be a topological space. A set $C \in \mathcal{P}(X)$ is closed iff there exists an open set $U \in \mathcal{U}$ such that $C = X \setminus U$.

3.1.14. **Example.** The empty set $\emptyset$ is closed. Let $(X,\mathcal{U})$ be a topological space. Since X is open and $X \setminus X = \emptyset$ then $\emptyset$ is closed. Recall that $\emptyset$ is also open — so closed does not mean "not open". Sets which are both closed and open are sometimes called *clopen*. Notice that X is also clopen since $X \setminus \emptyset = X$.

3.2. **The Importance of Being Hausdorff.** The Hausdorff property shows its relevance once we actually start to do some analysis so we will first define convergence in a topological setting:

3.2.1. **Definition.** *Convergent sequence.* A sequence $x_n \to x$ iff for every open set U with $x \in U$ there exists N such that $x_n \in U$ for all $n > N$. That is to say, the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x iff given any open set U with $x \in U$ then x_n is eventually always in U.

3.2.2. **Example.** $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ converges to 0 under the topology induced by $(\mathbb{R}, d_1)$. For any open set U in the topology, with $0 \in U$, there exists $B_\delta(0) \subseteq U$ for some $\delta > 0$. By the Archimedean property of $\mathbb{R}$ there exists N such that $0 < \frac{1}{n} < \delta$ for all $n > N$ so the sequence is eventually always in $B_\delta(0)$ which is in U.

3.2.3. **Definition.** *Hausdorff space.* A topological space $(X,\mathcal{U})$ is Hausdorff iff for every $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exist open sets $U_1, U_2 \in \mathcal{U}$ such that $U_1 \cap U_2 = \emptyset$

3.2.4. **Proposition.** *If a topological space is Hausdorff then limits of sequences are unique.*

Proof. Let $(X,\mathcal{U})$ be a Hausdorff topological space and let $x_n \to x_1$, $x_n \to x_2$. Suppose, for contradiction, that $x_1 \neq x_2$. Then by the definition of Hausdorff, there exist open sets

[1] The reader might wonder why we did not instead endow $\mathbb{S}^1$ with a subspace topology from $\mathbb{R}^2$ — we will later see in theorem 3.3.11 that we want the surrounding space to be "compact".

U_1, U_2 with $x \in U_1, x \in U_2$ such that $U_1 \cap U_2 = \emptyset$. Since $x_n \to x_1$ then by definition there is some N_1 such that $x_n \in U_1$ for all $n > N_1$. Likewise $x_n \to x_2$ implies there is some N_2 such that $x_n \in U_2$ for all $n > N_2$. Then $x_n \in U_1$ and $x_n \in U_2$ for all $n > \max(N_1, N_2)$. Therefore $U_1 \cap U_2 \neq \emptyset$ — contradiction. □

Recall that when we prove uniqueness of limits in $\mathbb{R}$, we usually take two candidates x_1 and x_2 and let the distance between them be $2\epsilon = |x_2 - x_1|$. Then we show that elements in the sequence cannot be less than ϵ away from both x_1 and x_2. This implicitly constructs the open balls $B_\epsilon(x_1)$ and $B_\epsilon(x_2)$ with empty intersection.

3.2.5. **Example.** The induced topology of (R, d_1) is Hausdorff. Take $x_1, x_2 \in \mathbb{R}$ with $x_1 \neq x_2$. Let $2\epsilon = |x_2 - x_1|$. Then $B_\epsilon(x_1) \cap B_\epsilon(x_2) = \emptyset$.

3.2.6. **Example.** The course topology on a set X containing more than 1 element is not Hausdorff. Recall example 3.1.4 with $X = \{\text{Alice}, \text{Bob}\}$ and the course topology $\mathcal{U} = \{\emptyset, \{\text{Alice}, \text{Bob}\}\}$. There are no open sets which contain Alice but not Bob. Then any sequence Alice, Bob, Alice, Alice, Bob, . . . converges to both Alice and Bob.

For a more subtle eaxmple of a non-Hausdorf topology see [Sut10, Ch. 7 Example 7.9, Ch. 11 Example 11.6].

The Hausdorff property is often included in definitions of other spaces such as a "smooth manifold" [Lee13, p. 3] to ward off undesirable behaviour such as limits not being unique. However, there are non-trivial classes of topologies which are naturally non-Hausdorff [DST19, p. 2].

3.2.7. **Definition.** *Continuous function.* Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be topological spaces. Let $f : X \to Y$ be a map between them. Then f is continuous iff $f^{-1}(V)$ is open in X whenever V is open in Y.

3.2.8. **Example.** $f(x, y) \mapsto x^2 + y^2$ is a continuous function between the product topology of $[-1, 1]^2$ (example 3.1.11) and the induced topology on $(\mathbb{R}, d_1)$. Denote the product topology of $[-1, 1]^2$ as $([-1, 1]^2, \mathcal{K})$ and the induced topology on $(\mathbb{R}, d_1)$ as $(\mathbb{R}, \mathcal{V})$. Let V be an open set in $\mathcal{V}$ and consider the point $z \in V$. We may assume $0 \leq z \leq 2$ as otherwise the pre-image of z is the empty set which is open in $[-1, 1]^2$ and $\emptyset \subseteq f^{-1}(V)$. There exists $B_\epsilon(z) \subseteq V$ for some $\epsilon > 0$. Consider $(x_0, y_0) \in f^{-1}(z)$ and the two open sets $B^{(1)}_\delta(x_0) := \{x \in [-1, 1] : |x_0 - x| < \delta\}$ and $B^{(2)}_\delta(y_0) := \{y \in [-1, 1] : |y_0 - y| < \delta\}$ where $\delta = \min\{1, \frac{\epsilon}{2(|x_0|+|y_0|)+2}\}$. Note that $B^{(1)}_\delta$ and $B^{(2)}_\delta$ are derived from the intersection of $[-1, 1]$ with true open balls in $(\mathbb{R}, d_1)$ (which are metric-open sets by example 2.0.10)

— so these intersections are open sets in the subspace topology on $[-1,1]$ (but technically not balls since we haven't defined them for subspaces).

Now for any $(x,y) \in B_\delta^{(1)}(x_0) \times B_\delta^{(2)}(y_0)$ then $|x-x_0| < \delta$. This implies $|x-x_0| < 1$ therefore $|x| - |x_0| \leq 1$ and $|x| \leq |x_0| + 1$ so $|x+x_0| \leq 2|x_0| + 1$. Similarly $|y-y_0| < \delta$ and $|y+y_0| \leq 2|y_0| + 1$. We now observe:

$$\begin{aligned} |f(x,y) - f(x_0,y_0)| &= |(x^2+y^2) - (x_0^2+y_0^2)| \\ &= |(x^2-x_0^2) + (y^2-y_0^2)| \\ &\leq |x^2-x_0^2| + |y^2-y_0^2| \\ &= |x-x_0|\,|x+x_0| + |y-y_0|\,|y+y_0| \\ &< \delta(2|x_0|+1) + \delta(2|y_0|+1) \leq \epsilon \end{aligned}$$

That is to say $f(B_\delta^{(1)}(x_0) \times B_\delta^{(2)}(y_0)) \subseteq B_\epsilon(z) \subseteq V$ so $B_\delta^{(1)}(x_0) \times B_\delta^{(2)}(y_0) \subseteq f^{-1}(V)$. Since this is true for any $z \in V$ we conclude $f^{-1}(V)$ is open for any open V in $(\mathbb{R}, \mathcal{U})$.

3.2.9. **Proposition.** *The pre-image of a closed set is also closed under a continuous function.*

Proof. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be topological spaces and $f: X \to Y$ a continuous function. For any closed set $D \in \mathcal{P}(Y)$ there exists $V \in \mathcal{V}$ such that $D = Y \setminus V$. Since f is continuous then $f^{-1}(V) = U$ is open in X. Therefore $C = f^{-1}(D) = X \setminus U$ is closed. □

3.2.10. **Example.** We see that $\mathbb{S}^1$ endowed with the subspace topology in 3.1.12 is closed. It is the pre-image of the closed set $[0,1]$ (in the induced topology from $(\mathbb{R}, d_1)$) under the continuous map in example 3.2.8.

3.3. **A Finite Discussion of Compactness.** Compactness is in some sense like finiteness but without requiring discreteness.

3.3.1. **Definition.** *Open cover.* Let $(X,\mathcal{U})$ be a topological space. An open cover of X is a subset $\mathcal{V} \subseteq \mathcal{U}$ such that $\bigcup_{V \in \mathcal{V}} V = X$. We say $\mathcal{V}$ *covers* X. If $\mathcal{W} \subseteq \mathcal{V}$ and $\mathcal{W}$ also covers X, then we say $\mathcal{W}$ is a *subcover* of X.

3.3.2. **Definition.** *Compact space.* A topological space $(X,\mathcal{U})$ is compact if every open cover of X has a finite subcover.

3.3.3. **Theorem.** (Heine-Borel pt.1) *The closed interval* $[-1,1]$ *is compact.*

Proof. Along the way we will use the method of continuous induction. Suppose $\mathcal{V}$ is an open cover of $[-1,1]$ and let $A = \{a \in [-1,1] : [-1,a] \text{ has a finite subcover}\}$. Since $\mathcal{V}$ covers $[-1,1]$ there is some $V_0 \in \mathcal{V}$ containing -1 so V_0 covers $[-1,-1]$. Therefore $-1 \in A$ i.e A is not empty.

Now consider $\alpha = \sup A$ which is well defined since A is bounded above and not empty. Suppose, for contradiction, that $\alpha < 1$. Then $\alpha \in [-1,1]$ and there is some $V_\alpha \in \mathcal{V}$ such that $\alpha \in V_\alpha$. Then there exists some $B_\delta(\alpha) \subseteq V_\alpha$. By definition of α as the supremum, we must have $\alpha - \frac{\delta}{2} \in A$ i.e. $[-1, \alpha - \frac{\delta}{2}]$ has a finite subcover $\mathcal{W}$. Take $\mathcal{W} \cup \{B_\delta(\alpha)\}$ and notice this is a finite subcover of $[-1, \min(1, \alpha + \frac{\delta}{2})]$ i.e. $\min(1, \alpha + \frac{\delta}{2}) \in A$ which contradicts the definition of α as the supremum of A. So $\sup(A) = 1$.

All that remains is to show $\sup(A) \in A$. There is some $V_1 \in \mathcal{V}$ with $1 \in V_1$ so there exists some $\epsilon > 0$ such that $B_\epsilon(1) \subseteq V_1$. Since $1 - \frac{\epsilon}{2} < \sup(A)$ then there exists a finite subcover $\mathcal{W}^*$ of $[-1, 1 - \frac{\epsilon}{2}]$. Then $\mathcal{W}^* \cup \{B_\epsilon(1)\}$ is a finite subcover of $[-1,1]$. □

Compactness suggests the space is in some sense "not infinitely large". In particular we shall see that a compact induced topology must come from a *bounded* metric space. But first we must define what it means to be a bounded metric space — which we omitted in section 2.

3.3.4. **Definition.** *Bounded metric space.* A metric space (X,d) is bounded iff there exists $r \in \mathbb{R}$ such that $d(x,y) \leq R$ for all $x, y \in X$.

3.3.5. **Example.** $(\mathbb{R}, d_1)$ is <u>not</u> bounded. For any $r \in \mathbb{R}$ we can find $n \in \mathbb{N}$ with $n > r$ (the Archimedean property of $\mathbb{R}$) so $d_1(0,n) = n > r$.

3.3.6. **Proposition.** *Every compact space is bounded.*

Proof. Let $(X,\mathcal{U})$ be a compact induced topology from the metric space (X,d). Take $x \in X$ then $\{B_r(x) : r > 0\}$ covers $\mathbb{R}$ since any other point $y \in X$ is some finite distance away. If $(X,\mathcal{U})$ is compact then there exists some finite subcover $\{B_{r_1}(x), \ldots, B_{r_n}(x)\}$. Let $r^* = \max\{r_1, \ldots, r_n\}$. Then $d(x,y) < r^*$ for all $y \in X$. Now for all $y, z \in X$ we have $d(y,z) \leq d(y,x) + d(x,z) < 2r^*$ so X is bounded. □

3.3.7. **Theorem.** (Heine-Borel pt.2) *The topology induced by* $(\mathbb{R}, d_1)$ *is* <u>*not*</u> *compact.*

Proof. By contrapositive. Since $(\mathbb{R}, d_1)$ is not bounded (example 3.3.5) then it's not compact, by proposition 3.3.6. □

3.3.8. **Proposition.** *If* $(X,\mathcal{U})$ *and* $(Y,\mathcal{V})$ *are both compact then the product topology* $(X \times Y, \mathcal{K}_{X\times Y})$ *is compact.*

Proof. We need to show that every open cover $\mathcal{L} \subseteq \mathcal{K}_{X\times Y}$ of $X \times Y$ has a finite subcover $\mathcal{T}$. First we consider the special case that every set L in $\mathcal{L}$ is of the form $L = U \times V$ where $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Since $\mathcal{L}$ is an open cover then for every $(x, y) \in X \times Y$ there exists $L_{x,y} \in \mathcal{L}$ with $(x, y) \in L_{x,y}$ and $L_{x,y} = U_{x,y} \times V_{x,y}$ with $U_{x,y} \in \mathcal{U}$ and $V_{x,y} \in \mathcal{V}$. Fix $x_0 \in X$. Then $V_{x_0} := \{V_{x_0,y} : y \in Y\}$ is an open cover of Y. Since Y is compact there exists a finite subcover $\{V_{x_0,y_1}, \dots, V_{x_0,y_m}\}$. Now take the finite intersection $U_{x_0} := \bigcap_{j=1}^{m} U_{x_0,y_j}$ which is open in X. Then $\mathcal{L}_{x_0} = \{L_{x_0,y_1}, \dots, L_{x_0,y_m}\}$ covers $U_{x_0} \times Y$ and $\{U_{x_0} : x_0 \in X\}$ is an open cover of X. Since X is compact, there exists a finite subcover $\{U_{x_1}, \dots, U_{x_n}\}$. Then $\mathcal{T} = \bigcup_{i=1}^{n} \mathcal{L}_{x_i} = \bigcup_{i=1,j=1}^{i=n,j=m} L_{x_i,y_j}$ is a finite subcover of $X \times Y$.

In the general case, suppose $\mathcal{L}'$ is an open cover of $X \times Y$. For every $(x, y) \in X \times Y$ there exists $L'_{x,y} \in \mathcal{L}'$ with $(x, y) \in L'_{x,y}$. Since $L'_{x,y}$ is open, there exists $U_{x,y} \in \mathcal{U}$ and $V_{x,y} \in \mathcal{V}$ with $x \in U_{x,y}$ and $y \in V_{x,y}$ such that $L_{x,y} = U_{x,y} \times V_{x,y} \subseteq L'_{x,y}$ with $(x, y) \in L_{x,y}$. Now $\{L_{x,y} : x \in X, y \in Y\}$ is an open cover of $X \times Y$ of the type discussed above. □

3.3.9. **Theorem.** (Heine-Borel pt.3) *The product topology on* $[-1, 1]^2$ *is compact.*

Proof. Theorem 3.3.3 tells us $[-1, 1]$ is compact as a subspace topology of the induced topology on $(\mathbb{R}, d_1)$. Proposition 3.3.8 tells us that the product topology of two compact topologies is also compact. □

3.3.10. **Proposition.** *If* $(X,\mathcal{U})$ *is a compact topology and* C *is closed in* $(X,\mathcal{U})$ *then the subspace topology* $(C, \mathcal{S})$ *is compact.*

Proof. Suppose $\mathcal{V} = V_\alpha : \alpha \in A$ is an open cover of C. By the definition of a subspace topology, for every $\alpha \in A$ there exist $V'_\alpha \in \mathcal{U}$ such that $V_\alpha = V'_\alpha \cap C$. Note that since $\mathcal{V}$ covers C then $C \subseteq \bigcup_{\alpha\in A} V'_\alpha$.

Since C is closed then $X\backslash C$ is open. So $\mathcal{V}' = \{V'_\alpha : \alpha \in A\} \cup X\backslash C$ is an open cover of X. Since X is compact, there exists a finite subcover $\mathcal{W}' = \{V'_{\alpha_1}, \dots, V'_{\alpha_n}\} \cup X\backslash C$. Since $C \cap X\backslash C = \emptyset$ then $\mathcal{W} = \{V_{\alpha_1}, \dots, V_{\alpha_n}\}$ is a finite subcover of C. □

3.3.11. **Theorem.** *The subspace topology* $(\mathbb{S}^1, \mathcal{S})$ *from* $[-1, 1]^2$ *is compact.*

Proof. Example 3.2.10 tell us that $\mathbb{S}^1$ is closed in $[-1, 1]^2$, example 3.3.9 tells us that $[-1, 1]^2$ is compact and proposition 3.3.10 tells us that a closed subspace of a compact topology forms a compact subspace topology. □

4. Conclusion

We need two more propositions to reach our final goal.

4.0.1. **Proposition.** *Any subspace topology of a Hausdorff space is also Hausdorff.*

Proof. Let $(X,\mathcal{U})$ be a Hausdorff topology and $(Y,\mathcal{S})$ a subspace topology with $Y \subseteq X$. Consider two distinct points $y_1, y_2 \in Y$. Since $Y \subseteq X$ then $y_1, y_2 \in X$. Since X is Hausdorff, then there exist open sets $U_1, U_2 \in \mathcal{U}$ with $y_1 \in U_1$ and $y_2 \in U_2$ such that $U_1 \cap U_2 = \emptyset$. Then $S_1 = U_1 \cap C$ and $S_2 = U_2 \cap C$ are open sets in $\mathcal{S}$ with $y_1 \in S_1$ and $Y_2 \in U_2$ such that $S_1 \cap S_2 = \emptyset$. □

4.0.2. **Proposition.** *Any product topology of two Hausdorff spaces is also Hausdorff.*

Proof. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be Hausdorff spaces. Let $(X \times Y, \mathcal{K}_{X\times Y})$ be their product topology. Consider two distinct points (x_1, y_1) and (x_2, y_2) in $X \times Y$. Then $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Since X and Y are both Hausdorff, there exist open sets $U_1, U_2 \in \mathcal{U}$ and $V_1, V_2 \in \mathcal{V}$ with $x_1 \in U_1$ and $x_2 \in U_2$ and $y_1 \in V_1$ and $y_2 \in V_2$ such that $U_1 \cap U_2 = \emptyset$ and $V_1 \cap V_2 = \emptyset$. Then $K_1 = U_1 \times V_1 \subseteq U_1 \times V_1$ and $K_2 = U_2 \times V_2 \subseteq U_2 \times V_2$ are open sets in $\mathcal{K}_{X\times Y}$ with $(x_1, y_1) \in K_1$ and $(x_2, y_2) \in K_2$ such that $K_1 \cap K_2 = \emptyset$. □

We showed in example 3.2.5 that the induced topology on (R, d_1) is Hausdorff. With proposition 4.0.1 we can show that the subspace topology on $[-1, 1]$ is therefore also Hausdorff. With proposition 4.0.2 we know that $[-1, 1]^2$ with the product topology is Hausdorff. Finally, we may apply proposition 4.0.1 again to show that $\mathbb{S}^1$ with the subspace topology from $[-1, 1]^2$ is Hausdorff. Combined with theorem 3.3.11 we conclude that $\mathbb{S}^1$ with the subspace topology of $[-1, 1]^2$ is a compact Hausdorff space.

Contrast this with example 3.2.5 and theorem 3.3.7 which tell us that the topology on $\mathbb{R}$ induced by $(\mathbb{R}, d_1)$ is a non-compact Hausdorff space.

The proof that the product topology of two compact spaces is also compact has been taken nearly ad verbum from notes written by Dexter Chua (based on the "Metric and Topological Spaces" course lectured in 2015 at Cambridge). Most other proofs are the author's own since the 16 page limit means that minimal ancillary material can be introduced — and almost all sources (including Chua's notes) leave the proof of many propositions as exercises.

At the time of writing, Dexter Chua's notes may be found at the Cambridge Student Run Computing Facility: https://dec41.user.srcf.net/notes/. These are an invaluable resource for any reader wanting a broader introduction to point-set topology (or indeed any other topic covered in a typical undergraduate mathematics degree).

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